Symmetric instability under full-component Coriolis force imposed by horizontal top and bottom boundary conditions

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1. Introduction

When the system governing the rotating fluids is non-hydrostatic, the horizontal component of the Coriolis force, which is omitted under the so-called "traditional approximation", is revived. Since this term causes unexpected effects on the fluid motion when boundaries are placed, for example, at the top and bottom of the domain, it is necessary to investigate sophisticatedly its role on the fluid motion. Here, we focused on the symmetric motion, i.e. the transversal motion in the sheared zonal flow under both ambient rotation and stratification, and try to analyze the particular motion caused by the top and bottom boundaries under full Coriolis force.

2. Governing equations

The Boussinesq equations linearized around the zonal flow $U(y,z)$ which has both y- and z-, i.e. northward and vertical, directions are used for the analysis. By introducing the stream function Ψ, defined to be $v' = \partial \Psi / \partial z$, $w' = \partial \Psi / \partial y$ where v' and w' indicate the northward and vertical components of the perturbation velocity, respectively, they can be combined into a single equation as follows:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi + F^2\frac{\partial^2\psi}{\partial z^2} + 2S^2\frac{\partial^2\psi}{\partial y\partial z} + N'^2\frac{\partial^2\psi}{\partial y^2} = 0$$

$$F^2 = f_V\left(f_V - \frac{\partial U}{\partial y}\right),\quad S^2 = f_V\left(f_H + \frac{\partial U}{\partial z}\right),\quad N'^2 = N^2 + f_H\left(f_H + \frac{\partial U}{\partial z}\right) \tag{1}$$

where t denotes time, N the buoyancy frequency, and f_V and f_H indicate vertical and horizontal component of the planetary vorticity, respectively.

3. Solution under the top and bottom rigid boundaries

Imposing rigid boundary conditions at the top and bottom of the domain, i.e. $\Psi=0$ at $z=0$ and H, the solution of (1) is obtained as follows:

$$\psi \propto \sin\left(n\frac{\pi}{H}z\right)\exp\left[i\left\{l\left(y + \frac{S^2}{\omega_\pm^2 - F^2}z\right) - \omega_\pm t\right\}\right]. \tag{2}$$

$$\omega_{\pm}^2 = F^2 + \frac{1}{2}(N'^2 - F^2)\sin^2\chi' \pm \sqrt{[\frac{1}{2}(N'^2 - F^2)\sin^2\chi']^2 + S^4\sin^2\chi'} \qquad (3)$$

where $\sin\chi' = 1/[(n\pi/H)^2 + l^2]^{1/2}$ and $n=1,2,3,\cdots$. Different from the unbounded case, where the dispersion relation is quadratic of the angular frequency ω (Itano and Maruyama, 2009), the quartic equation on ω and the corresponding four modes are obtained in this bounded case as already reported in the studies on IGW under such slantwise ambient rotation(Thuburn et al., 2002; Kasahara, 2003) and the symmetric instability under the traditional approximation(Xu, 2007).

4. Properties of the eigen modes and the discriminant of the symmetric instability

By differentiating (3) with $|\sin\chi'|$, we get

$$\frac{\partial\omega_{\pm}^2}{\partial|\sin\chi'|} = \frac{\sqrt{L^2 - 4S^8} \pm L}{\sqrt{L + 2S^4}} \begin{cases} > 0 \ (for\ \omega_{+}^2) \\ < 0 \ (for\ \omega_{-}^2) \end{cases} \qquad (4)$$

where $L=(N'^2 - F^2)\sin^2\chi' + 2S^4 \ (\geqq 0)$. Thus, ω_{+}^2 (ω_{-}^2) is a monotonically increasing (decreasing) function of $|\sin\chi'|$ where its maximum and minimum occur at $|\sin\chi'| = 1(0)$ and $0(1)$, respectively. From (4), the following inequality is obtained:

$$\omega_{min}^2 \leq \omega_{-}^2 < F^2 < \omega_{+}^2 < \omega_{max}^2 \qquad (5)$$

where

$$\omega_{\substack{max\\min}}^2 = \frac{1}{2}[N'^2 + F^2 \pm \sqrt{(N'^2 + F^2)^2 + 4D}], \qquad D = S^4 - F^2 N'^2. \qquad (6)$$

The inequality (5) indicates a couple of high frequency modes($\pm\omega_{+}$) is super-inertial, and that of low frequency modes($\pm\omega_{-}$) is sub-inertial. Meanwhile, according to (6), when D is positive, ω_{min}^2 becomes negative so that the low frequency modes could be unstable. Therefore, D is proven to be the discriminant of the symmetric instability. Note that ω_{max}^2, ω_{min}^2 and D are identical to those in the unbounded case.

5. Essential dimensionless numbers

Two of three dimensionless numbers defined as the ratio of three parameters appeared in (1), i.e. $(F/S)^2$, $(N'/F)^2$, $(S/N')^2$ completely determine the fundamental nature of the symmetric motion and stability regardless of the boundary conditions and the existence of f_H. Here, $(F/S)^2$ indicates the elevation angle of the constant momentum surface and N'/F the geometrical ratio of the frequency of the gravity wave to the frequency of the inertial wave in the case of including both vertical and horizontal shears of the basic flow.It is possible to construct a stability diagram with such two dimensionless numbers.