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Development of High-precision Nonhydrostatic Atmospheric Model (1): Governing Equations

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Synopsis

For atmospheric sciences in the 21st century, atmospheric numerical models with high precision are indispensable tools to be an exact science. Development of non-hydrostatic models based on fully compressible dynamic equations in particular are given high priority because they use the least approximation in dynamics. In this paper, we examined accuracy of finite difference expressions of equations used in non-hydrostatic fully compressible models. It is found that to make a set of flux form equations is difficult under the constraint of satisfaction of hydrostatic balance with high precision. It is also found the set of equations we propose here are not fully flux form but have good conservation characteristics.

Keywords : nonhydrostatic model, equations, flux form, conservation

1. Introduction

Synchronizing with rapid development of computer science and technology, numerical models in meso- to small scale meteorology uses less approximations to the governing equations recently. Now, most mesoscale models uses compressible equations as their dynamics, while many cloud models uses anelastic equations about fifteen years ago.

Compressible models have several merits. One of the merits concerns Poisson solvers for pressure. Over complex terrain, Poisson solvers for anelastic (or Boussinesq) equation systems in terrain following coordinates have been consuming a big amount of computer resources. However, pressure or its substitute variable is one of prognostic variables in compressible models. Therefore, the solver of Poisson equation for pressure are no longer an essential part of the model.

There are demerits of compressible models. One of the demerits is that the time increments must be small enough to satisfy the Courant-Friedrichs-Lewy condition for sound waves. However, numerical technique such as a time-splitting method or a full- (or semi-) implicit method can alleviate the problem of the short time increment. The recent increase of computer power also supports the migration from anelastic to compressible models.

There are several variations of governing dynamic equations among compressible numerical models in meteorology. One of the big differences is whether the model uses fully compressible (without any approximations) equations (e. g., Satomura, 1989; Saito, 1997) or a quasi- compressible (with one or more approximations) equations (e. g., Klemp and Wilhelmson, 1978; Dudhia, 1993; Xue et al., 1995). Another difference is whether the equations are in flux forms or in advective forms. In this paper, we focus on the latter difference among fully compressible models and examine the accuracy under the constraint of hydrostatic balance.

2. Equations

We examine the following five forms of two-dimensional fully compressible nonhydrostatic dynamical equations. Here, we define as $U = \rho u, W = \rho w, \Theta' = (\rho\theta)'$.

• *Advection form:*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + c_p\theta\frac{\partial \pi'}{\partial x} = 0 \quad (1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + c_p\theta\frac{\partial \pi'}{\partial z} = -g\frac{\theta'}{\Theta} \quad (2)$$

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z} = 0 \quad (3)$$

$$\frac{\partial \pi'}{\partial t} + u\frac{\partial \pi'}{\partial x} + w\frac{\partial \rho pi'}{\partial z} - \frac{gw}{c_p\theta} = \frac{R\pi}{c_v\theta}\frac{d\theta'}{dt} - \frac{R\pi}{c_v}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right) \quad (4)$$

This form was used by, for example, Tapp and White (1977), Carpenter (1979), Satomura (1989).

• *Flux form 1:*

$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial wU}{\partial z} + \gamma R\pi\frac{\partial \Theta'}{\partial x} = 0 \quad (5)$$

$$\frac{\partial W}{\partial t} + \frac{\partial uW}{\partial x} + \frac{\partial wW}{\partial z} + \gamma R\pi\frac{\partial \Theta'}{\partial z} = -g\left(\bar{\rho}\frac{\pi'}{\bar{\pi}} - \rho'\right) \quad (6)$$

$$\frac{\partial \Theta'}{\partial t} + \frac{\partial U\Theta'}{\partial x} + \frac{\partial W\Theta'}{\partial z} = 0 \quad (7)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0 \quad (8)$$

This form was used in WRF model (Klemp et al., 2000).

• *Flux form 2:*

$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial wU}{\partial z} + \frac{\partial p'}{\partial x} = 0 \quad (9)$$

$$\frac{\partial W}{\partial t} + \frac{\partial uW}{\partial x} + \frac{\partial wW}{\partial z} + \frac{\partial p'}{\partial z} = -\rho' g \quad (10)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0 \quad (11)$$

$$\begin{cases} \frac{\partial \rho e}{\partial t} + \frac{\partial Uh}{\partial x} + \frac{\partial Wh}{\partial z} \\ \quad = \left(u\frac{\partial p'}{\partial x} + w\frac{\partial p'}{\partial z} + \rho' wg\right) - \rho wg \\ \frac{\partial p'}{\partial t} + \frac{\partial c_s^2 W}{\partial z} = \frac{R_d}{c_v}\left(G_E - W\tilde{g}\right) \end{cases} \quad (12)$$

where

$$\tilde{g} = g - \frac{1}{\rho}\left(\frac{\partial p'}{\partial z} + \rho' g\right),$$

$$G_E = -\frac{\partial Uh}{\partial x} + u\frac{\partial p'}{\partial x}.$$

This form was developed by Satoh (2002). In this equation set, both of the equations in (12) are used simultaneously in the model time integration to guarantee the energy conservation.

• *Quasi-flux form 1:*

$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial wU}{\partial z} + \frac{\partial p'}{\partial x} = 0 \quad (13)$$

$$\frac{\partial W}{\partial t} + \frac{\partial uW}{\partial x} + \frac{\partial wW}{\partial z} + \frac{\partial p'}{\partial z} = -g\rho' \quad (14)$$

$$\frac{\partial \theta'}{\partial t} + \frac{1}{\rho}\frac{\partial \theta' U}{\partial x} + \frac{1}{\rho}\frac{\partial \theta W}{\partial z} = \frac{\theta}{\rho}\left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z}\right) \quad (15)$$

$$\frac{\partial p'}{\partial t} + \frac{c_p R\theta}{c_v}\left(\frac{p}{p_0}\right)^{R/c_p} \times \left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} - \frac{\rho}{\theta}\frac{\partial \theta'}{\partial t}\right) = 0 \quad (16)$$

This form was used in MRI/JMA-NHM (Saito et al. 2001)

• *Quasi-flux form 2:*

$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial wU}{\partial z} + c_p\rho\theta\frac{\partial \pi'}{\partial x} = 0 \quad (17)$$

$$\frac{\partial W}{\partial t} + \frac{\partial uW}{\partial x} + \frac{\partial wW}{\partial z} + c_p\rho\theta\frac{\partial \pi'}{\partial z} = -g\rho\frac{\theta'}{\Theta} \quad (18)$$

$$\frac{\partial \pi'}{\partial t} + u\frac{\partial \pi'}{\partial x} + w\frac{\partial \pi}{\partial z} = \frac{R\pi}{c_v\theta}\frac{d\theta'}{dt} - \frac{R\pi}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \quad (19)$$

$$\frac{\partial \theta'}{\partial t} + u\frac{\partial \theta'}{\partial x} + w\frac{\partial \theta'}{\partial z} = 0 \quad (20)$$

• *Quasi-flux form 3:*

$$\frac{\partial U}{\partial t}+\frac{\partial uU}{\partial x}+\frac{\partial wU}{\partial z}+\frac{\partial p'}{\partial x}=0 \qquad (21)$$

$$\frac{\partial W}{\partial t}+\frac{\partial uW}{\partial x}+\frac{\partial wW}{\partial z}+\frac{\partial p'}{\partial z}=-g\rho' \qquad (22)$$

$$\frac{\partial p'}{\partial t}+\frac{c_pR}{c_vp_0}\left(\frac{p}{p_0}\right)^{R/c_p} \times\left(\frac{\partial U\theta'}{\partial x}+\frac{\partial W\theta}{\partial z}\right)=0 \qquad (23)$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}=0 \qquad (24)$$

This form was found by authors and firstly used by Akiba (2002) in her master thesis.

3. Hydrostatic balance

In this section, accuracy of hydrostatic balance of above equation sets is examined. Hydrostatic balance is one of the most important balance for the atmosphere of the earth. Most of all atmospheric motions are driven by small amount of imbalance from the hydrostatic balance. If the model code cannot reproduce the hydrostatic balance precisely, unrealistic flow will appear in the mode. Therefore, it is important for nonhydrostatic models to use equations which assure basic hydrostatic balance.

Let's start to examine the equation set "flux form 1". This set of equations has π' in the right-hand-side of eq. (6). This variable is calculated as

$$\begin{aligned}\pi' &= \pi-\overline{\pi}\\ &= \left(\frac{R_d\Theta}{p_0}\right)^{R_d/c_v}-\left(\frac{R_d\bar{\Theta}}{p_0}\right)^{R_d/c_v}\end{aligned} \qquad (25)$$

Normally, ^pi is nearly equal to $\overline{\pi}$ and, therefore, such kind of subtraction possibly causes a serious cancellation error. This fact indicates that one must carefully make a discrete form of the equation to avoid the cancellation error.

The set "flux form 2" does not have such a possible cancellation error. This set, however, uses the energy conservation equation in different forms eq. (19) two times. As Satoh (2002) also described in his paper, twice use of same equation in different forms introduces a kind of inconsistency, while he stated this inconsistency seemed to be small.

The set "quasi-flux form 1" also has the cancellation error problem. In this set, ρ' is not a prognostic variable but is calculated as

$$\rho'=\rho-\bar{\rho}$$

Similar to the calculation of π' in the set "flux form 1", ρ is nearly equal to $\bar{\rho}$ and special attention should be paid to make a discrete form of the equation set.

The other sets "advection form", "quasi-flux form 2" and "quasi-flux form 3" do not suffer from the cancellation error due to the subtraction of hydrostatic variable ($\overline{\pi}$ or $\bar{\rho}$) from its total variable (π or ρ). In this sense, these tree forms are superior to the "full flux form 1", "full flux form 2" and "quasi-flux form 1".

4. Mass conservation

Another point to be considered is the aspect of the energy conservation. Generally, finite difference schemes which are not full flux forms are difficult to conserve the total energy. To examine how much energy will be lost or added by time integration, a heat island experiment was conducted. In this paper, results using the "quasi-flux form 2" and "quasi-flux form 3" were shown. The experiment conditions were:

- the centered both in time and space finite difference schemes were used for the Arakawa A grid system.
- the grid increment was 200 m both in the horizontal and vertical directions,
- the domain size was 80 km in the horizontal and 10 km in the vertical direction,
- both the horizontal and the vertical boundaries were rigid wall,
- the horizontal and the top boundaries were free slip, and the bottom boundary was the viscous boundary,
- the right half of the bottom surface was 5 K higher than the left half,
- one-equation turbulent closure scheme was used.

Figure 1 shows the time change of total mass for the "quasi-flux form 2". The mass monotonically increased with time. The increase rate of mass was

$$\begin{aligned}\frac{1}{M}\frac{dM}{dt} &\approx \frac{1}{8\times 10^4}\frac{60}{45\times 3600}\\ &= 4.6\times 10^{-9} \quad \mathrm{sec}^{-1}.\end{aligned} \qquad (26)$$

This mass increase rate indicates that the average pressure will increase about 1 hPa after 2

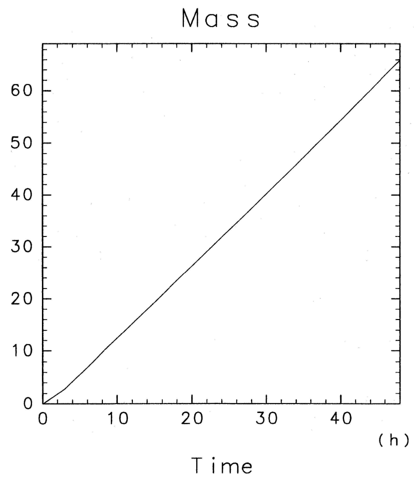

Fig. 1 Change of total mass with time for the "quasi-flux form 2". The unit of the ordinate is kg.

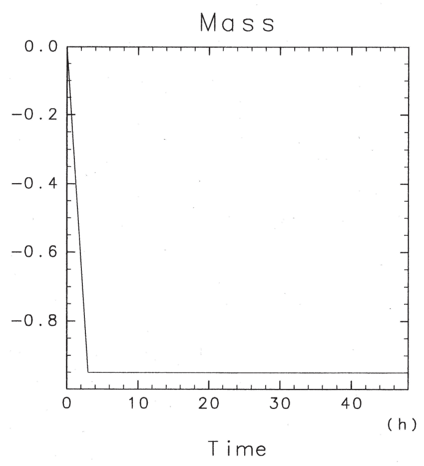

Fig. 2 Same as Fig. 1 except for the "quasi-flux form 3".

days integration. This is a rather large change of mass for climate models.

Figure 2 shows the time change of total mass for the "quasi-flux form 3". In this case, the mass initially decreased only about 0.9 kg and then kept almost the constant value.

Figures 3 and 4 show the potential temperature and the horizontal velocity after 6 hours integration using the "quasi-flux form 3". It is shown that a mixed layer was formed over the heated surface up to the 1.2 km and convective motions dominated in the mixed layer. The flow toward the heated surface existed below 400   500 m height, and the reverse flow was observed above. Therefore, the simulation using the "quasi-flux form 3"

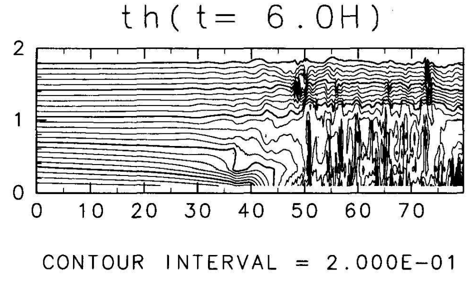

Fig. 3 Potential temperature after 6 hours integration for the "quasi-flux form 3". Contour interval is 0.2 K.

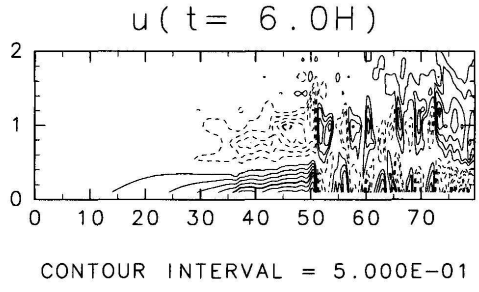

Fig. 4 Same as Fig. 3 except horizontal velocity. Contour interval is 0.5 $m\ s^{-1}$.

not only conserved the total mass, but also captured the characteristics of the heat island circulation well.

5. Conclusion

Six forms of two-dimensional nonhydrostatic fully compressible hydrodynamic equations were compared from the view point of the hydrostatic balance. It was found that tree forms named "full flux form 1" used in WRF model (Klemp et al., 2000), "full flux form 2" (Satoh, 2002) and "quasi-flux form 1" used in MRI/JMA-NHM (Saito et al., 2001) possibly suffered from the cancellation error by subtracting a value nearly equal to the other value. Other three forms "advection form" (e. g., Tapp and White, 1977; Carpenter, 1979; Satomura, 1989), "quasi-flux form 2" and "quasi-flux form 3" found by authors were free from such the cancellation error.

Mass conservation was also examined for the "quasi-flux form 2" and "quasi-flux form 3". By simulating a heat island circulation, it is found that the total mass of "quasi-flux form 2" monotonically increased while the total mass of "quasi-flux form 3" was almost constant during the simulation.

As a conclusion, the "quasi-flux form 3" is best form from the view points of numerical calculation error and mass conservation among the six forms examined in this paper. Probably the "quasi-flux form 3" can be used in climate simulation models in the future because of its less numerical error and good mass conservation characteristics.

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高精度非静力学大気モデルの開発
（1）方程式系

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要　旨

精密科学を指向する 21 世紀の気象学にとって，高精度の大気モデルは必要不可欠の道具である。特に，対象現象のスケールに制限のない完全圧縮流体力学方程式に基づく非静力学モデルの構築への要請は高く，既に幾つかの試みが発表されている。本研究では，完全圧縮非静水圧流体力学方程式系のいくつかの表現形式の差分化を行ったときの精度について検討した。その結果，静水圧平衡を精度良く満たしつつ完全なフラックス形式の差分方程式をつくることは，状態方程式の非線型性によって困難であること，完全なフラックス形式ではなくても保存性の優れたものがあることが見いだされた。

キーワード： 非静力学，方程式系，フラックス形式，保存性