NOTES AND CORRESPONDENCE

Supplementary Note on Shear Instability in a Shallow Water

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1. Introduction

In a previous paper (Satomura, 1981; hereafter referred to as I), the stability problem of two types of shear flows was studied. One type of shear flow was the plane Couette flow bounded in both sides by two rigid walls, and the other was the same flow but is unbounded in one side to connect to a rest fluid which extends to infinity. One of the purposes of I was to clarify the second mode instability which was first found by Blumen *et al.* (1975). Since they used a hyperbolic tangent profile of a compressible fluid as a basic shear flow, the physical mechanism of destabilization for the second mode was not clear owing to the existence of a point of inflection. A conclusion of I is that those waves are gravity waves (or acoustic waves in the case of a compressible fluid) destabilized by the basic shear flow. The other conclusion is that, in essence, the unstable waves do not arise from the existence of an inflection point but they are destabilized by the linear shear under the presence of divergence. The other purpose of I was to examine possibility of generation of gravity waves which radiate from the shear zone. The possibility was demonstrated.

Since the basic flows examined in I are bounded in one or two sides, the comparison of the results of I with Drazin and Davey (1977), which is an extension of Blumen *et al.* (1975), is not straightforward. In the present note, we will obtain eigenvalues for a piecewise-linear shear flow unbounded in both sides, which is regarded as an approximation of a hyperbolic tangent flow, and compare them with the results of Drazin and Davey (1977).

2. Eigenvalue problem

We consider a stability of a piecewise-linear plane parallel flow with a velocity

$$U(y)=\begin{cases}1 & 1\leq y\\ y & 0\leq y\leq 1\,,\\ 0 & y\leq 0\end{cases} \qquad (1)$$

in a shallow water. Then it follows (cf. I) that the stability of this basic flow is governed by the equations

$$\frac{d^2h}{dy^2}+k^2\{F_r^2(1-C)^2-1\}h=0 \quad \text{for} \quad 1\leq y\,, \qquad (2)$$

$$(y-C)\frac{d^2h}{dy^2}-2\frac{dh}{dy}-k^2\{Fr^2(y-C)^2-1\}h=0 \quad \text{for } 0\leq y\leq 1\,, \qquad (3)$$

$$\frac{d^2h}{dy^2}+k^2\{F_r^2C^2-1\}h=0 \quad \text{for } y\leq 0\,. \qquad (4)$$

The boundary conditions are that h is finite or radiating from the shear zone to $y\rightarrow\infty$. The interfacial boundary conditions are that the velocity component normal to the internal boundary, v, and the surface displacement, h, should be continuous at $y=0$ and $y=1$. Here we use dimensionless variables throughout, and assume that the perturbation surface displacement has the form as $h(y)\exp\{ik(x-Ct)\}$ in terms of the positive wavenumber k and complex phase speed $C=C_r+iC_i$. The Froude number Fr is defined as the ratio of the characteristic velocity of the basic flow U_0 to the phase speed of gravity wave $\sqrt{gH}$, where H is the mean depth of the shallow water.

As in I, general solution of (3) is expressed as

$$h=A\sum_{n=3\ \text{odd}}^{\infty}a_n(y-C)^n-B\sum_{n=0\ (\text{even})}^{\infty}a_n(y-C)^n\,, \qquad (4)$$

where A and B are constants and a's defined by (4.13) of I. Solutions of (2) and (4) which satisfy the boundary conditions at $|y|\rightarrow\infty$ are written as

$$h=A_+e^{\beta_+y} \quad \text{for } 1\leq y\,, \qquad (5)$$

$$h=A_-e^{\beta_-y} \quad \text{for } y\leq 0\,, \qquad (6)$$

where A_+ and A_- are constants and

$$\beta_+=-k\{1-F_r^2(1-C)^2\}^{1/2}\,, \qquad (7)$$

$$\beta_-=k\{1-F_r^2C^2\}^{1/2}\,. \qquad (8)$$

We choose branches of square roots in the right-hand sides of (7) and (8) so as to make real parts of β_+ and β_- non-positive and non-negative, respectively. These branches of square roots also satisfy the radiation condition where $|y|\to\infty$. Then the characteristic equation is deduced from the requirement that all constants, A, B, A_+ and A_- do not vanish when solutions (4)–(6) satisfy interfacial boundary conditions. The method of solving the characteristic equation and the accuracy of the eigenvalues are the same as those in I.

3. Results

Figs. 1 and 2 display distributions of phase speed ΔC_r and growth rate kC_i on the k-Fr plane, respectively, where $\Delta C_r=|0.5-C_r|$. Figs. 3–7 show graphs of ΔC_r and kC_i as a function of wavenumber k for five typical values of Fr, respectively.

From these figures it is easily found that there are two different types of unstable waves; one has a phase speed $C_r=0.5$ and the largest growth rate at $Fr=0$, *i.e.*, non-divergent limit (mode I). The other appears only for $Fr>1$ and its phase speed C_r is not 0.5 (mode II). Obviously, the former type is associated with an inflection point (barotropic instability), and the latter can exist by the effect of divergence and basic shear flow. Mode I has two different eigenvalues at the same k and Fr for $1.8\lesssim Fr\lesssim 2.7$. For $Fr\lesssim 1.8$ mode I is single-valued and for $2.7\lesssim Fr$ it disappears. It is worth noting that propagating neutral modes exist for $Fr\lesssim 1.8$, and, from Fig. 5, a propagating neutral mode exist at $Fr\sim 1.8$ even for smaller wavenumber than the cut off wavenumber of mode I.

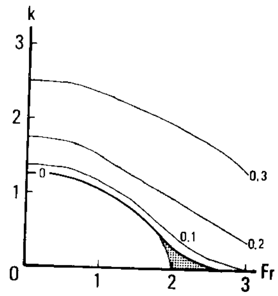

Fig. 1 Difference of phase speed from 0.5, *i.e.*, $\Delta C_r=|0.5-C_r|$. Multivalued unstable region is indicated by shade. A stationary unstable mode ($\Delta C_r=0$ and $C_i=0$) exists in a region between a thick solid line $\Delta C_r=0$ and the Fr axis and outside the dotted region. The thin solid lines depict isolines of ΔC_r.

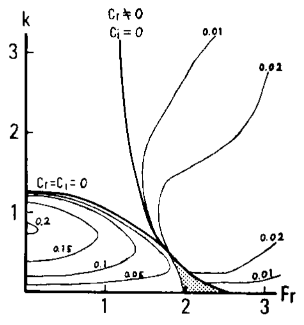

Fig. 2 Growth rate kC_i. Multi-valued unstable region is indicated by shade. The thin solid lines depict isolines of kCi. Thick solid line depicts $C_i=0$.

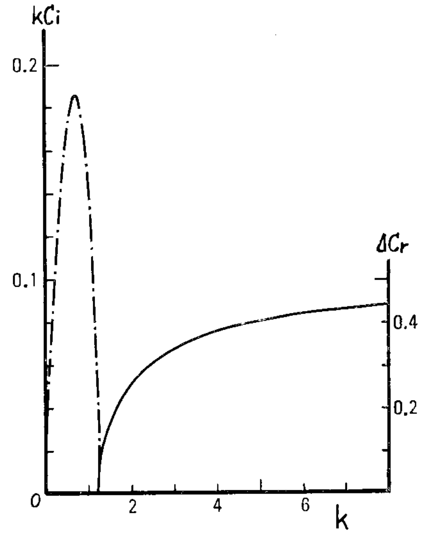

Fig. 3 Graphs of $\Delta C_r=|0.5-C_r|$ and kC_i for $Fr=0.5$. Solid line depicts ΔC_r. This mode of $\Delta C_r\neq 0$ is neutral. Chain line denotes growth rate kC_i of stationary mode ($\Delta C_r=0$).

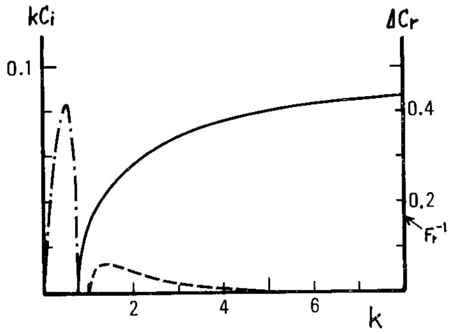

Fig. 4 Same as Fig. 3 except for $Fr=1.5$. Dashed line depicts growth rate kC_i of travelling mode ($\Delta C_r\neq 0$).

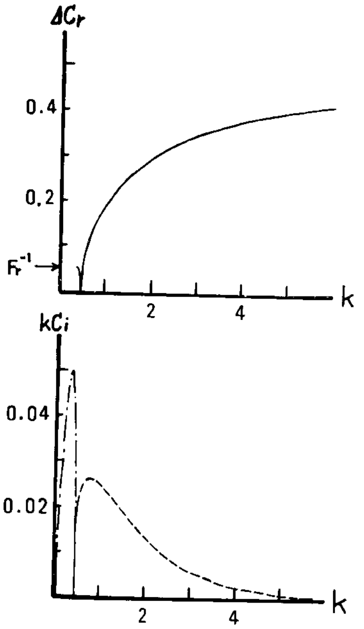

Fig. 5 Same as Fig. 4 except for $Fr=1.8$.

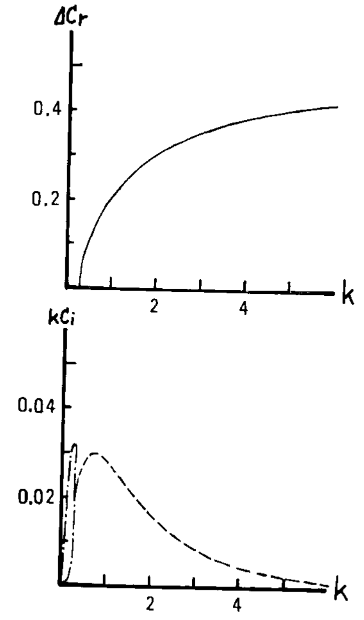

Fig. 6 Same as Fig. 4 except for $Fr=2.0$.

Fig. 8 shows structures of mode I and II. This figure indicates that mode I is quickly damped away from shear zone and mode II propagates from one side of shear zone to infinity. Of course, mode II contains a pair of waves which have the same growth rate but whose phase speeds differ from 0.5 by the same value, *i.e.*, $C_r=0.5\pm\Delta C_r$. Since the solutions in the rest fluids have a form as (5) and (6), the wave which has a faster phase speed $C_r=0.5+\Delta C_r$ is sinusoidal for $y\leq 0$ and exponential for $y\geq 1$. For the wave of slower phase speed $C_r=0.5-\Delta C_r$, its property is reversed.

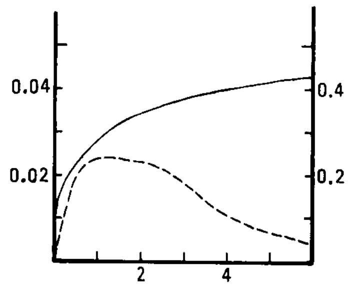

Fig. 7 Same as Fig. 4 except for $Fr=3.0$.

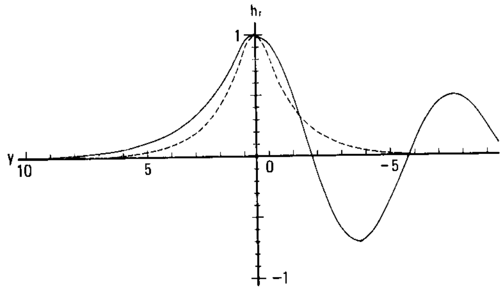

Fig. 8 Real part of eigenfunction h for stationary mode and travelling mode. Solid line depicts travelling mode: $Fr=2.2$, $k=0.7$, $\Delta C_r=0.183$, $kC_i=0.029$, $\beta_+=-0.510-0.061i$, $\beta_-=0.085-0.788i$. Dashed line depicts stationary mode: $Fr=0.5$, $k=0.75$, $\Delta C_r=0$, $kC_i=0.184$, $\beta_\pm=\mp 0.732-0.024i$.

When we compare Figs. 1–7 with Figs. 1–3 of Drazin *et al.* (1977) and Fig. 1 of Blumen *et al.* (1975)*, we notice some similarities and differences. They are summarized as follows:

a) Dependence of growth rate of mode I on k and Fr is similar to that in the case of hyperbolic tangent flow. It is reasonable, because mode I exists only for small k and piecewise-linear flow (1) is a good approximation of $U=\tanh(y)$ for small k as Esch (1957) showed.

b) Striking characteristics such as mode bifurcation and multivalued nature of C which were found by Drazin *et al.* (1977) are also found near $Fr=2$ in our shear flow.

* Note that our definitions of nondimensional wave-number and Froude number differ from those of Drazin *et al.* (1977) and Blumen *et al.* (1975) by a factor two.

c) In the case of (1), non-singular neutral modes are continuous to unstable mode and they exist for, at least, finite range of k-C_r plane. When Fr exceeds 1, larger wavenumber part becomes unstable. Thus 'high-wave number cutoff' does not appear for mode II. In the case of hyperbolic tangent flow, singular neutral mode (see Miles, 1961), which is continuous to unstable non-singular mode, is not continuous to any other singular neutral modes as Yih (1974) suggested. Instead of the singular neutral mode, only the continuous-spectrum solution may exist outside the marginally stable curve in k-Fr (or k-M) plane as commented by Drazin *et al.* (1977). By evaluating (4.20′) of I, we confirmed the fact that, in the case of hyperbolic tangent flow, no normal-mode eigen-solution exists outside the unstable domain in k-Fr plane (not shown).

The reason why the behaviour of eigen-solution differs from that of Drazin *et al.* (1977) in a region of large wavenumber of k-Fr plane is understood as follows: The equation (7) of Blumen (1970) is transformed to a normal form

$$(\bar{u}-c)\frac{d^2f}{dy^2}=-[(\bar{u}-c)\bar{u}''-2\bar{u}'^2 -\alpha^2(\bar{u}-c)^2\{1-M^2(\bar{u}-c)^2\}]f \qquad (9)$$

where $f=\hat{\pi}/(\bar{u}-c)$ and other symbols are the same as Blumen (1970). In the case of $\bar{u}=\tanh(y)$, the first and the second terms of the right-hand side of (9) can be neglected comparing the last term when wavenumber α is very large and $\bar{u}=c$. Then a simple WKB solution of (9) shows that the effect of the shear only changes the 'wavenumber' in y direction and neutral normal mode remains neutral, if it exists. In the case of the piecewise-linear flow, however, $\bar{u}''$ becomes infinitely large (delta function) at $y=0$ and $y=1$. Thus, the first term in the right-hand side of (9) cannot be neglected. It means that the effect of shear (or the effect of abrupt change of shear at interfaces) is important even when wavenumber becomes very large. In this case, neutral normal-mode solutions have a possibility to be $C_i \neq 0$ and in fact they are unstable for $Fr>1$. Physically, it is rather obvious. The scale of the wave in y direction, say L_y, might be small if the scale in x direction is small. Thus, smoothly varying basic flow will vary only slightly in one 'wavelength' by L_y and waves may not be affected by the basic shear remarkably. But in the case of piecewise-linear flow, even if L_y becomes very small, waves may feel abrupt change of the basic shear and may be destabilized.

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浅水のシヤー不安定に関する補遺

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