

a. (14.5)

$$\begin{aligned}\rho \frac{d\mathbf{v}}{dt} &= -\nabla p - \rho g \mathbf{k} \\ \nabla \cdot \mathbf{v} &= 0\end{aligned}$$

に対し,

$O(\epsilon^0)$:

$$\begin{aligned}0 &= -\nabla \bar{p} - \rho g \mathbf{k} = -\frac{\partial \bar{p}}{\partial z} - \rho g \\ \nabla \cdot \bar{\mathbf{v}} &= 0\end{aligned}$$

$O(\epsilon^1)$:

$$\begin{aligned}\rho \frac{D\mathbf{v}'}{Dt} &= -\nabla p' \quad \left(\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \\ \nabla \cdot \mathbf{v}' &= 0\end{aligned}$$

$O(\epsilon^0)$ と $O(\epsilon^1)$ を元の方程式から減じて,

$$\begin{aligned}\rho \left[\frac{\partial \bar{\mathbf{v}}}{\partial t} + (\mathbf{v}' \cdot \nabla)(\bar{\mathbf{v}} + \mathbf{v}') \right] &= 0 \\ \frac{\partial \bar{\mathbf{v}}}{\partial t} &= -(\mathbf{v}' \cdot \nabla)(\bar{\mathbf{v}} + \mathbf{v}')\end{aligned}$$

zonal mean をとり u 成分だけを考えると,

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial t} &= -\overline{\mathbf{v}' \cdot \nabla(\bar{u} + u')} \\ &= -\overline{\mathbf{v}' \cdot \nabla u'} \\ &= -\left(\overline{u' \frac{\partial u'}{\partial x}} + \overline{v' \frac{\partial u'}{\partial y}} + \overline{w' \frac{\partial u'}{\partial z}} \right) \\ &= -\left[\frac{\partial \overline{(u'u')}}{\partial x} + \frac{\partial \overline{(v'u')}}{\partial y} + \frac{\partial \overline{(w'u')}}{\partial z} - \overline{u' \nabla \cdot \mathbf{v}'} \right] \\ &= -\frac{\partial \overline{(v'u')}}{\partial y} - \frac{\partial \overline{(w'u')}}{\partial z} \quad \left(\frac{\partial}{\partial x}(\text{zonal mean}) = 0 \right)\end{aligned}$$

b. (14.13) *Parseval's theorem*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} F(k) e^{ikx} dk \right|^2 dx$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} F(k) e^{ikx} dk \right] \left[\int_{-\infty}^{\infty} F^*(k') e^{-ik'x} dk' \right] dx \\
&= \int_{-\infty}^{\infty} F(k) dk \int_{-\infty}^{\infty} F^*(k') dk' \int_{-\infty}^{\infty} e^{-i(k'-k)x} dx \\
&= \int_{-\infty}^{\infty} F(k) dk \int_{-\infty}^{\infty} F^*(k') dk' \cdot \delta(k'-k) \\
&= \int_{-\infty}^{\infty} F(k) dk \cdot F^*(k) \\
&= \int_{-\infty}^{\infty} |F(k)|^2 dk
\end{aligned}$$

即ち,

$$\|f(x)\| = \|f(k)\|$$

函数列 $\{1, \sin kx, \cos kx\}$ が完備であることから明らか.

c. (14.14)

$$\begin{aligned}
\iiint_{-\infty}^{\infty} \frac{\partial p'}{\partial x} e^{-i(kx+ly-\sigma t)} dx dy dt &= \iint_{-\infty}^{\infty} \left\{ [p' e^{-ikx}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p' (-ik) e^{-ikx} dx \right\} dy dt \\
&= \iint_{-\infty}^{\infty} \left\{ 0 + ik \int_{-\infty}^{\infty} p' e^{-ikx} dx \right\} dy dt \\
&= ik \iiint_{-\infty}^{\infty} p' e^{-i(kx+ly-\sigma t)} dx dy dt \\
&= ik P_{kl}^{\sigma}
\end{aligned}$$

(p' は $(-\infty, \infty)$ で絶対可積な函数なので $p' \rightarrow 0$ as $x \rightarrow \pm\infty$)

d. (14.22.1)

$$\eta_k(k) = \begin{cases} A & k = k \pm dk \\ A^* & k = -k \pm dk \\ 0 & \text{elsewhere} \end{cases}$$

に対して,

$$\begin{aligned}
\eta'_k(x, t) &= \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \eta_k(k') \cdot e^{i(k'x-\sigma't)} dk' d\sigma' \\
&= \frac{1}{4\pi^2} \left[\eta_k(k+dk) e^{i[(k+dk)x-(\sigma+d\sigma)t]} + \eta_k(k-dk) e^{i[(k-dk)x-(\sigma-d\sigma)t]} \right. \\
&\quad \left. + \eta_k(-k+dk) e^{i[(-k+dk)x-(\sigma+d\sigma)t]} + \eta_k(-k-dk) e^{i[(-k-dk)x-(\sigma-d\sigma)t]} \right] \\
&= \frac{A}{4\pi^2} \left[e^{i[(k+dk)x-(\sigma+d\sigma)t]} + e^{i[(k-dk)x-(\sigma-d\sigma)t]} \right] \\
&\quad + \frac{A^*}{4\pi^2} \left[e^{i[(-k+dk)x-(\sigma+d\sigma)t]} + e^{i[(-k-dk)x-(\sigma-d\sigma)t]} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{A}{4\pi^2} e^{i(kx-\sigma t)} \left[e^{i(dk \cdot x - d\sigma \cdot t)} + e^{-i(dk \cdot x - d\sigma \cdot t)} \right] \\
&\quad + \frac{A^*}{4\pi^2} e^{-i(kx-\sigma t)} \left[e^{i(dk \cdot x - d\sigma \cdot t)} + e^{-i(dk \cdot x - d\sigma \cdot t)} \right] \\
&= \frac{1}{4\pi^2} \left[A e^{i(kx-\sigma t)} + A^* e^{-i(kx-\sigma t)} \right] \left[e^{i(dk \cdot x - d\sigma \cdot t)} + e^{-i(dk \cdot x - d\sigma \cdot t)} \right] \\
&= \frac{1}{4\pi^2} \left[A e^{i(kx-\sigma t)} + (A e^{i(kx-\sigma t)})^* \right] \cdot 2 \cos(dk \cdot x - d\sigma \cdot t) \\
&= \frac{1}{2\pi^2} \cdot 2 \operatorname{Re} \left[A e^{i(kx-\sigma t)} \right] \cos(dk \cdot x - d\sigma \cdot t) \\
&= \frac{1}{\pi^2} \operatorname{Re} \left[A e^{i(kx-\sigma t)} \right] \cos(dk \cdot x - d\sigma \cdot t)
\end{aligned}$$

$A \in \mathbb{R}$ なら, $A \equiv \pi^2$ ととって,

$$= \cos(dk \cdot x - d\sigma \cdot t) \cos(kx - \sigma t)$$

e.

$x \in \mathbb{R}, z \in \mathbb{C}$ とする.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(x+iu)^2} dx &= \lim_{L \rightarrow +\infty} \int_{-L}^L e^{-(x+iu)^2} dx \\
&= \lim_{L \rightarrow +\infty} \int_{-L+iu}^{L+iu} e^{-x'^2} dx'
\end{aligned}$$

正則関数 e^{-z^2} に対し,

$$\oint e^{-z^2} dz = \left[\int_{-L+iu}^{L+iu} + \int_{L+iu}^L + \int_L^{-L} + \int_{-L}^{-L+iu} \right] e^{-z^2} dz = 0$$

$$\begin{aligned}
J_1 &= \int_{L+iu}^L e^{-z^2} dz \\
&= i \int_u^0 e^{-(L+iy)^2} dy \\
&= i e^{-L^2} \int_u^0 e^{y^2 - 2iLy} dy \\
|J_1| &\leq e^{-L^2} \int_u^0 e^{y^2} dy \rightarrow 0 \quad \text{as } L \rightarrow +\infty \\
J_2 &= \int_{-L}^{-L+iu} e^{-z^2} dz \\
&= - \int_L^{L-iu} e^{-z^2} dz \\
&= \int_{L-iu}^L e^{-z^2} dz
\end{aligned}$$

$$\begin{aligned}
&= i \int_{-u}^0 e^{-(L+iy)^2} dy \\
&= ie^{-L^2} \int_{-u}^0 e^{y^2-2iLy} dy \\
|J_2| &\leq e^{-L^2} \int_{-u}^0 e^{y^2} dy \rightarrow 0 \quad \text{as } L \rightarrow +\infty
\end{aligned}$$

故,

$$\begin{aligned}
\lim_{L \rightarrow +\infty} \int_{-L+iu}^{L+iu} e^{-z^2} dz &= - \lim_{L \rightarrow +\infty} \left[\int_{L+iu}^L + \int_L^{-L} + \int_{-L}^{-L+iu} \right] e^{-z^2} dz \\
&= - \lim_{L \rightarrow +\infty} \left[0 + \int_L^{-L} + 0 \right] e^{-z^2} dz \\
&= \lim_{L \rightarrow +\infty} \int_{-L}^L e^{-z^2} dz \\
&= \lim_{L \rightarrow +\infty} \int_{-L}^L e^{-x^2} dx \\
&= \int_{-\infty}^{\infty} e^{-x^2} dx \\
&= \sqrt{\pi}
\end{aligned}$$

f. below (14.23.1) (Problem 14.8)

$$\begin{aligned}
\eta_k(k) &= \frac{1}{\sqrt{2\pi} dk} e^{-\frac{(k-k_0)^2}{2dk^2}} \\
\eta'_k(x) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} dk} e^{-\frac{(k'-k_0)^2}{2dk^2}} \cdot e^{ik'x} dk' \\
&= \frac{1}{(\sqrt{2\pi})^5 dk} \int_{-\infty}^{\infty} e^{-\frac{1}{2dk^2} [(k'-k_0)^2 - 2dk^2 ik'x]} dk' \\
&= \frac{1}{(\sqrt{2\pi})^5 dk} \int_{-\infty}^{\infty} e^{-\frac{1}{2dk^2} [k'^2 - 2(idk^2 x + k_0)k' + k_0^2]} dk' \\
&= \frac{1}{(\sqrt{2\pi})^5 dk} \int_{-\infty}^{\infty} e^{-\frac{1}{2dk^2} [\{k' - (idk^2 x + k_0)\}^2 - (idk^2 x + k_0)^2 + k_0^2]} dk' \\
&= \frac{1}{(\sqrt{2\pi})^5 dk} e^{-\frac{1}{2dk^2} [-(idk^2 x + k_0)^2 + k_0^2]} \int_{-\infty}^{\infty} e^{-\frac{1}{2dk^2} \{(k'+k_0) - idk^2 x\}^2} dk' \\
&= \frac{1}{(\sqrt{2\pi})^5 dk} e^{-\frac{1}{2dk^2} (dk^4 x^2 - 2ik_0 dk^2 x)} \int_{-\infty}^{\infty} e^{-\frac{1}{2dk^2} (k'+k_0)^2} dk' \\
&= \frac{1}{(\sqrt{2\pi})^5 dk} e^{\left(-\frac{dk^2}{2} x^2 + ik_0 x\right)} \cdot \sqrt{2dk^2 \pi} \\
&= \frac{1}{(2\pi)^2} e^{-\frac{dk^2}{2} x^2} \cdot e^{ik_0 x}
\end{aligned}$$

g. (14.24)

$$\begin{aligned}
\omega &= [g|\mathbf{k}| \tanh(|\mathbf{k}|H)]^{\frac{1}{2}} \\
c_g = \frac{\partial \omega}{\partial |\mathbf{k}|} &= \frac{1}{2} [g|\mathbf{k}| \tanh(|\mathbf{k}|H)]^{-\frac{1}{2}} \cdot g \left[\tanh(|\mathbf{k}|H) + |\mathbf{k}|H \frac{1}{\cosh^2(|\mathbf{k}|H)} \right] \\
&= \frac{1}{2} \sqrt{gH} [|\mathbf{k}|H \tanh(|\mathbf{k}|H)]^{-\frac{1}{2}} \cdot \left[\tanh(|\mathbf{k}|H) + |\mathbf{k}|H \frac{1}{\cosh^2(|\mathbf{k}|H)} \right] \\
&= \frac{1}{2} \sqrt{gH} \left[\frac{\tanh(|\mathbf{k}|H)}{|\mathbf{k}|H} \right]^{-\frac{1}{2}} \cdot \left[\frac{\tanh(|\mathbf{k}|H)}{|\mathbf{k}|H} + \frac{1}{\cosh^2(|\mathbf{k}|H)} \right] \\
&\rightarrow \frac{1}{2} \sqrt{gH} \cdot 1 \cdot 2 \quad \text{as } |\mathbf{k}|H \rightarrow 0 \\
&= \sqrt{gH} \\
&= c
\end{aligned}$$

h. (14.25.2)

$$\begin{aligned}
\eta'(x, y, 0) &= \frac{1}{(\sqrt{2\pi L})^2} e^{-\frac{x^2+y^2}{2L^2}} \\
\eta_{kl} &= \int \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi L})^2} e^{-\frac{x^2+y^2}{2L^2}} \cdot e^{-i(kx+ly)} dx dy \\
&= \frac{1}{2\pi L^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}(x^2+i2L^2kx)} dx \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}(y^2+i2L^2ly)} dy \\
&= \frac{1}{2\pi L^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}[(x+iL^2k)^2+L^4k^2]} dx \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}[(y+iL^2l)^2+L^4l^2]} dy \\
&= \frac{1}{2\pi L^2} e^{-\frac{L^2}{2}k^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}(x+iL^2k)^2} dx \cdot e^{-\frac{L^2}{2}l^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}(y+iL^2l)^2} dy \\
&= \frac{1}{2\pi L^2} e^{-\frac{L^2}{2}(k^2+l^2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}x^2} dx \cdot e^{-\frac{L^2}{2}l^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2L^2}y^2} dy \\
&= \frac{1}{2\pi L^2} e^{-\frac{L^2}{2}(k^2+l^2)} \cdot \sqrt{2L^2\pi} \cdot \sqrt{2L^2\pi} \\
&= e^{-\frac{L^2}{2}(k^2+l^2)} \\
|\eta_{kl}|^2 &= e^{-L^2(k^2+l^2)}
\end{aligned}$$

i. (14.26)

$$\begin{aligned}
\omega &= [g|\mathbf{k}| \tanh(|\mathbf{k}|H)]^{\frac{1}{2}} \\
c_g = \frac{\partial \omega}{\partial |\mathbf{k}|} &= \frac{1}{2} [g|\mathbf{k}| \tanh(|\mathbf{k}|H)]^{-\frac{1}{2}} \cdot g \left[\tanh(|\mathbf{k}|H) + |\mathbf{k}|H \frac{1}{\cosh^2(|\mathbf{k}|H)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sqrt{g} [|\mathbf{k}| \tanh(|\mathbf{k}|H)]^{-\frac{1}{2}} \cdot \left[\tanh(|\mathbf{k}|H) + |\mathbf{k}|H \frac{1}{\cosh^2(|\mathbf{k}|H)} \right] \\
&\rightarrow \frac{1}{2} \sqrt{g} \cdot (|\mathbf{k}| \cdot 1)^{-\frac{1}{2}} \cdot (1 + 0) \quad \text{as } |\mathbf{k}|H \rightarrow \infty \\
&= \frac{1}{2} \sqrt{\frac{g}{|\mathbf{k}|}} \\
&= \frac{c}{2}
\end{aligned}$$

j. (14.33.2)

$$\begin{aligned}
-gH \frac{\partial \hat{p}'}{\partial z} + g\hat{p}' &= -RT \frac{\partial}{\partial z} \left(\frac{p'}{\bar{p}} \right) + g\hat{p}' \\
&= -RT \left(\frac{1}{\bar{p}} \frac{\partial p'}{\partial z} - p' \cdot \frac{1}{\bar{p}^2} \frac{\partial \bar{p}}{\partial z} \right) + g\hat{p}' \\
&= -RT \left(\frac{1}{\bar{p}} \frac{\partial p'}{\partial z} - \frac{\hat{p}'}{\bar{p}} \frac{\partial \bar{p}}{\partial z} \right) + g\hat{p}' \\
&= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} + \hat{p}' \cdot \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + g\hat{p}' \\
&= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z} + \hat{p}' \cdot (-g) + g\hat{p}' \\
&= -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial z}
\end{aligned}$$

k. (14.33.3)

$$\begin{aligned}
\frac{1}{\rho} \frac{d\rho}{dt} &\sim \frac{1}{\bar{\rho}} (1 - \hat{\rho}') \left(\frac{D\bar{\rho}}{Dt} + \frac{D\rho'}{Dt} + u' \frac{\partial \bar{\rho}}{\partial x} + w' \frac{\partial \bar{\rho}}{\partial z} \right) \\
&\sim \frac{1}{\bar{\rho}} \left(\frac{D\bar{\rho}}{Dt} + \frac{D\rho'}{Dt} + w' \frac{\partial \bar{\rho}}{\partial z} \right) - \frac{\hat{\rho}'}{\bar{\rho}} \frac{D\bar{\rho}}{Dt} \quad (\bar{\rho} = \bar{\rho}(z)) \\
&= \frac{1}{\bar{\rho}} \frac{D\bar{\rho}}{Dt} + \left(\frac{1}{\bar{\rho}} \frac{D\rho'}{Dt} - \rho' \cdot \frac{1}{\bar{\rho}^2} \frac{D\bar{\rho}}{Dt} \right) + \frac{1}{\bar{\rho}} \cdot w' \frac{\partial \bar{\rho}}{\partial z} \\
&= \frac{1}{\bar{\rho}} \frac{D\bar{\rho}}{Dt} + \frac{D\hat{\rho}'}{Dt} - \frac{w'}{H} \\
w' \frac{\partial \bar{\rho}}{\partial z} &= w' \frac{\partial}{\partial z} \left(\frac{\bar{p}}{RT} \right) \\
&= \frac{w'}{RT} \frac{\partial \bar{p}}{\partial z} \\
&= \frac{w'}{RT} \cdot (-\bar{\rho}g)
\end{aligned}$$

$$= -\bar{\rho} \frac{w'}{H}$$

1. (14.37) (14.40)

波動解を仮定:

$$\begin{pmatrix} u' \\ w' \\ \hat{p}' \end{pmatrix} = \begin{pmatrix} U_k^\sigma \\ W_k^\sigma \\ P_k^\sigma \end{pmatrix} e^{i(kx - \sigma t)}$$

(14.37.1) に代入:

$$-i\omega U_k^\sigma + \frac{\partial \bar{u}}{\partial z} W_k^\sigma = -gH \cdot ik P_k^\sigma \quad (1.1)$$

(14.37.3) に代入:

$$ik U_k^\sigma + \frac{\partial W_k^\sigma}{\partial z} = 0 \quad (1.2)$$

(14.37.4) に代入:

$$\left[(-i\omega)^2 + N^2\right] W_k^\sigma + gH \cdot (-i\omega) \frac{\partial P_k^\sigma}{\partial z} = 0 \quad (1.3)$$

(1.1) $\times k$ + (1.2) $\times \omega$:

$$\begin{aligned} k \frac{\partial \bar{u}}{\partial z} W_k^\sigma + \omega \frac{\partial W_k^\sigma}{\partial z} &= -gH \cdot ik^2 P_k^\sigma \\ gH P_k^\sigma &= -\frac{1}{ik^2} \left[\omega \frac{\partial W_k^\sigma}{\partial z} + k \frac{\partial \bar{u}}{\partial z} W_k^\sigma \right] \\ gH P_k^\sigma &= \frac{i}{k} \left[(c_x - \bar{u}) \frac{\partial W_k^\sigma}{\partial z} + \frac{\partial \bar{u}}{\partial z} W_k^\sigma \right] \\ gH \frac{\partial P_k^\sigma}{\partial z} &= \frac{i}{k} \left[-\frac{\partial \bar{u}}{\partial z} \frac{\partial W_k^\sigma}{\partial z} + (c_x - \bar{u}) \frac{\partial^2 W_k^\sigma}{\partial z^2} + \frac{\partial^2 \bar{u}}{\partial z^2} W_k^\sigma + \frac{\partial \bar{u}}{\partial z} \frac{\partial W_k^\sigma}{\partial z} \right] \\ &= \frac{i}{k} \left[(c_x - \bar{u}) \frac{\partial^2 W_k^\sigma}{\partial z^2} + \frac{\partial^2 \bar{u}}{\partial z^2} W_k^\sigma \right] \quad (14.39) \end{aligned}$$

(1.3) and (14.39):

$$\begin{aligned} (N^2 - \omega^2) W_k^\sigma - i\omega \frac{i}{k} \left[(c_x - \bar{u}) \frac{\partial^2 W_k^\sigma}{\partial z^2} + \frac{\partial^2 \bar{u}}{\partial z^2} W_k^\sigma \right] &= 0 \\ (N^2 - \omega^2) W_k^\sigma + (c_x - \bar{u}) \left[(c_x - \bar{u}) \frac{\partial^2 W_k^\sigma}{\partial z^2} + \frac{\partial^2 \bar{u}}{\partial z^2} W_k^\sigma \right] &= 0 \\ \frac{\partial^2 W_k^\sigma}{\partial z^2} + \left[\frac{N^2 - \omega^2}{(c_x - \bar{u})^2} + \frac{\frac{\partial^2 \bar{u}}{\partial z^2}}{(c_x - \bar{u})} \right] W_k^\sigma &= 0 \quad (14.40.1') \end{aligned}$$

$$m^2(z) = \frac{N^2}{(c_x - \bar{u})^2} - k^2 + \frac{\partial^2 \bar{u}}{\partial z^2} \quad (14.40.2)$$

と置いて (14.40) を得る.

修正履歴:

- 2001/11/07: a.: $u' \nabla \cdot \mathbf{v}'$ $\overline{u' \nabla \cdot \mathbf{v}'}$
 b.: $e^{ik'x}$ $e^{-ik'x}$, $(k - k')$ $(k' - k)$
 l.: 追加.
 その他雑多な変更.